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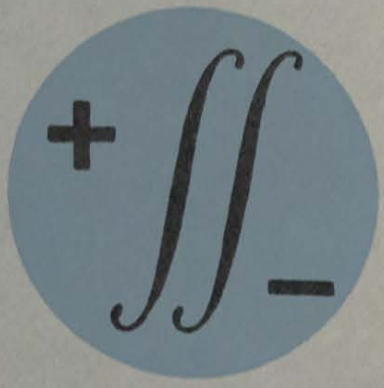

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On Creep and Relaxation

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The creep function for the "National Bureau of Standards" polyisobutylene at 25°C is calculated from the relaxation function, using the integral equation connecting them. The calculated function is shown to be correct in its initial slope, to satisfy the inequality $\varphi(t)\psi(t)\leqq 1$ throughout, and to approach satisfactorily the independently determined asymptote.

THE mathematical theory of linear viscoelasticity has contained integral equations of the convolution type ever since the formulation of the superposition principle by Boltzmann[1] in 1876. While such equations succinctly and elegantly express the relation between various functions, their utility has been slight, either because of the difficulty of expressing known functions in analytical form, or, even if this can be done, because of the difficulty of solving the resulting equations.

An example of such an equation is that connecting relaxation and creep functions. This has been variously expressed; the following is possibly the simplest representation of it. The creep and relaxation functions, $\varphi(t)$ and $\psi(t)$ respectively, may be defined as follows:

$$\varphi(t)=\frac{\text{strain at time } t}{\text{strain at time zero}}$$

after the instantaneous application of a fixed stress at time zero, and

$$\psi(t)=\frac{\text{stress at time } t}{\text{stress at time zero}}$$

after the instantaneous application of a fixed strain at time zero.

Allowing for the change in nomenclature and definition of the functions, it may easily be shown that the general expression by Gross (reference 1, p. 47) for the relation between them reduces to

$$\int_0^t \varphi(t-\tau)\psi(\tau)d\tau=t \tag{1}$$

or the equivalent

$$\int_0^t \varphi(\tau)\psi(t-\tau)d\tau=t. \tag{2}$$

The following sections will show that it is possible to solve Eqs. (1) and (2) by numerical means for one function if the other is given in tabular form. Specifically, the relaxation data for the "National Bureau of Standards" polyethylene, as given by Tobolsky and Catsiff[2] will be used in Eq. (2) for calculation of the creep function. Since $\psi(t)$ values normalized to $\psi(0)=1$ were necessary, they were calculated from the given $\log E_{r,298}(t)$ data by use of the following expression:

$$\psi(t)=10^{(\log E_{r,298}(t)-\log E_{r,298}(0))}$$
$$=10^{(\log E_{r,298}(t)-10.48)}.$$

THE NUMERICAL SOLUTION OF EQ. (2)

We first introduce the integral of $\psi(t)$

$$f(t)=\int_0^t \psi(\tau)d\tau.$$

Thus $f(0)=0$ and $f'(\tau)=\psi(\tau)$.

Next we introduce the notation $t_i(i=0,\cdots,n)$ with $t_0=0$, $t_1=10^{-14.6}$, $t_2=10^{-14.4}$, etc., since these are the times for which the $\psi(t)$ function is given, and *assume* $\psi(t)$ is a constant between t_0 and t_1.

In order to calculate $f(t)$ we use the trapezoid rule for integration,

$$f(t_{n+1})=f(t_n)+\tfrac{1}{2}[\psi(t_{n+1})+\psi(t_n)][t_{n+1}-t_n]$$

since the data do not justify any more elaborate method.

We now have from Eq. (2),

$$t_{n+1}=\int_0^{t_{n+1}} \varphi(\tau)\psi(t_{n+1}-\tau)d\tau$$
$$=\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}} \varphi(\tau)\psi(t_{n+1}-\tau)d\tau.$$

In each of the integrals in the sum we make the approximation

$$\int_{t_i}^{t_{i+1}} \varphi(\tau)\psi(t_{n+1}-\tau)d\tau=\varphi(t_{i+\frac{1}{2}})\int_{t_i}^{t_{i+1}} f'(t_{n+1}-\tau)d\tau$$
$$=-\varphi(t_{i+\frac{1}{2}})[f(t_{n+1}-t_{i+1})-f(t_{n+1}-t_i)],$$

[1] L. Boltzmann, Ann. d. Physik Ergänzungsband 7, 624 (1876). Other examples may be found in H. Leaderman *Elastic and Creep Properties of Filamentous Materials and Other High Polymers* (Textile Foundation, Washington, D. C., 1943), and B. Gross, *Mathematical Structure of the Theories of Viscoelasticity* (Hermann & Cie., Paris, 1953).

[2] A. V. Tobolsky and E. Catsiff, J. Polymer Sci. 19, 111 (1956).

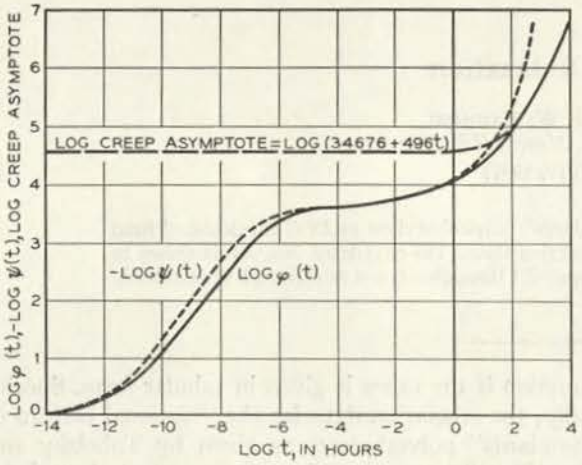

FIG. 1. Relaxation and creep functions and creep asymptote for National Bureau of Standards polyisobutylene at 25°C.

where $t_{i+\frac{1}{2}}$ is the midvalue $\frac{1}{2}(t_{i+1}+t_i)$. Thus

$$t_{n+1}=-\sum_{i=0}^{n-1}\varphi(t_{i+\frac{1}{2}})[f(t_{n+1}-t_{i+1}) - f(t_{n+1}-t_i)]+\varphi(t_{n+\frac{1}{2}})f(t_{n+1}-t_n).$$

Solving for $\varphi(t_{n+\frac{1}{2}})$ we have

$$\varphi(t_{n+\frac{1}{2}}) = \frac{t_{n+1}-\sum_{i=0}^{n-1}\varphi(t_{i+\frac{1}{2}})[f(t_{n+1}-t_i)-f(t_{n+1}-t_{i+1})]}{f(t_{n+1}-t_n)}. \quad (3)$$

This provides a method for calculating each value $\varphi(t_{n+\frac{1}{2}})$ in turn, where the first value is given by

$$\varphi_{\frac{1}{2}}=\frac{t_1}{f(t_1)}. \quad (4)$$

The creep function $\varphi(t)$ as calculated from Eqs. (3) and (4) is given in Table I and Fig. 1.

The reason Eqs. (3) and (4) are successful, and a number of other ways of approximating the integral are not, is that an error in calculating a $\varphi(t)$ (due to faulty data or even just numerical roundoff of a product) does not grow in size in subsequent stages of the computation. To see this intuitively we calculate the coefficient of $\varphi(t_{n-\frac{1}{2}})$ in the expression for $\varphi(t_{n+\frac{1}{2}})$. This coefficient is

$$-\frac{f(t_{n+1}-t_{n-1})-f(t_{n+1}-t_n)}{f(t_{n+1}-t_n)-f(t_{n+1}-t_{n+1})}.$$

Apply the mean value theorem to both the numerator and the denominator separately.

$$-\frac{\psi(t_{n+1}-\theta_1)(t_n-t_{n-1})}{\psi(t_{n+1}-\theta_2)(t_{n+1}-t_n)}=-\left[\frac{\psi(t_{n+1}-\theta_1)}{\psi(t_{n+1}-\theta_2)}\right]\left[\frac{t_n-t_{n-1}}{t_{n+1}-t_n}\right]$$

where $t_n>\theta_1>t_{n-1}$, $t_{n+1}>\theta_2>t_n$. Each bracket is less than 1 in magnitude so that the error in $\varphi(t_{n-\frac{1}{2}})$ is multiplied by a number much less than 1 in size. Thus the effect of the error oscillates and dies out rapidly.

CONFIRMATION OF CALCULATION

There are three tests which may be applied to confirm the consistency of $\varphi(t)$ with function $\psi(t)$ from which it was calculated. The first of these is that the slope of $\varphi(t)$ must be the negative of the slope of $\psi(t)$ at $t=0$. That this should be true is well known, and easily confirmed by substituting

$$\varphi(t)=1+a_1t+a_2t^2\cdots$$

and

$$\psi(t)=1-b_1t+b_2t^2\cdots$$

TABLE I. Stress relaxation $\psi(t)$ and creep $\varphi(t)$ for N.B.S. polyisobutylene at 25°C as functions of the time t. Note that the value of the log creep function in any row is not be be associated with the value of log time in the same row, but rather to the value of log time intermediate between that in the same row and that in the next.

log$t$	$-\log\psi(t)$	$\log\varphi(1.26t)$[a]	log$t$	$-\log\psi(t)$	$\log\varphi(1.26t)$[a]
−14.4	0	0	0.6	3.52	3.51
0.2	0	0.01	0.4	3.54	3.53
−14.0	0.02	0.02	0.2	3.56	3.55
−13.8	0.03	0.03	−5.0	3.57	3.57
0.6	0.04	0.04	−4.8	3.58	3.58
0.4	0.05	0.06	0.6	3.58	3.58
0.2	0.07	0.08	0.4	3.58	3.59
−13.0	0.09	0.09	0.2	3.59	3.59
−12.8	0.11	0.12	−4.0	3.60	3.60
0.6	0.14	0.15	−3.8	3.60	3.60
0.4	0.18	0.19	0.6	3.61	3.61
0.2	0.23	0.23	0.4	3.61	3.62
−12.0	0.28	0.27	0.2	3.63	3.63
−11.8	0.33	0.33	−3.0	3.64	3.65
0.6	0.41	0.35	−2.8	3.66	3.66
0.4	0.50	0.46	0.6	3.67	3.68
0.2	0.60	0.54	0.4	3.69	3.69
−11.0	0.71	0.63	0.2	3.70	3.71
−10.8	0.83	0.73	−2.0	3.73	3.73
0.6	0.96	0.83	−1.8	3.75	3.75
0.4	1.09	0.94	0.6	3.77	3.78
0.2	1.22	1.05	0.4	3.80	3.80
−10.0	1.36	1.16	0.2	3.83	3.83
−9.8	1.49	1.28	−1.0	3.86	3.86
0.6	1.62	1.40	−0.8	3.89	3.89
0.4	1.75	1.52	0.6	3.93	3.93
0.2	1.88	1.65	0.4	3.98	3.98
−9.0	2.01	1.77	0.2	4.03	4.02
−8.8	2.15	1.90	0.0	4.09	4.08
0.6	2.29	2.03	0.2	4.15	4.13
0.4	2.43	2.16	0.4	4.22	4.20
0.2	2.56	2.29	0.6	4.31	4.27
−8.0	2.68	2.42	0.8	4.40	4.35
−7.8	2.79	2.54	1.0	4.51	4.43
0.6	2.90	2.66	0.2	4.63	4.53
0.4	3.00	2.78	0.4	4.78	4.64
0.2	3.10	2.90	0.6	4.94	4.75
−7.0	3.19	3.01	0.8	5.14	4.88
−6.8	3.27	3.12	2.0	5.30	5.01
0.6	3.34	3.21	0.2	5.58	5.16
0.4	3.40	3.30	0.4	5.98	...
0.2	3.44	3.37	0.6	6.48	...
−6.0	3.48	3.43	0.8	∞	
−5.8	3.50	3.47			

[a] Since $10^{\log t+0.1}=1.26t$.

in Eq. (1) or (2), integrating, and letting t approach zero. At small values of t, $\log\varphi(t)\approx a_1 t/2.303$ and $\log\psi(t)\approx -b_1 t/2.303$, whence $\log\varphi(t)\approx -\log\psi(t)$. Table I shows that this condition obtains near the origin within the available accuracy.

The second test is suggested by the relation[3] $\varphi(t)\psi(t)\leqq 1$, whence $\log\varphi(t)\leqq -\log\psi(t)$, both terms in this inequality being positive. Table I and Fig. 1 show that this requirement is met.

The third test is provided by the independently calculable creep asymptote. In the case of a system with a finite maximum relaxation time, in which the ultimate value of $\psi(t)$ therefore approaches zero, the position as well as the slope of the creep asymptote may be found if either $\psi(t)$, the relaxation function, or $G(\tau)$, the relaxation time distribution, is known. Assume that the upper relaxation time is T_1, and let $\varphi(t)\simeq a+bt$ be the asymptote. Then for a time $t>T\gg T_1$, Eq. (1) may be written

$$\int_0^T \varphi(t-\tau)\psi(\tau)d\tau = t. \tag{5}$$

Taking the derivative with respect to t,

$$\int_0^T \varphi'(t-\tau)\psi(\tau)d\tau = 1.$$

Then

$$\varphi'(t-\tau)=\text{const}=b=1\Big/\int_0^T \psi(\tau)d\tau. \tag{6}$$

Using Eq. (5) with $\varphi\equiv a+bt$

$$\int_0^T [a+b(t-\tau)]\psi(\tau)d\tau = t$$

whence, by integration and substitution of Eq. (6),

$$a=b^2\int_0^T \tau\psi(\tau)d\tau. \tag{7}$$

These expressions for a and b are in terms of the relaxation function $\psi(\tau)$; but since, by definition of $\psi(t)$, and again assigning T_1 as the upper value of τ

$$\psi(t)=\int_0^{T_1} G(\tau)e^{-t/\tau}d\tau\Big/\int_0^{T_1} G(\tau)d\tau,$$

$$\int_0^t \psi(t)dt=\int_0^t\int_0^{T_1} G(\tau)e^{-t/\tau}d\tau dt\Big/\int_0^{T_1} G(\tau)d\tau$$

$$=\frac{1}{\int_0^{T_1} G(\tau)d\tau}\int_0^{T_1}[1-e^{-t/\tau}]\tau G(\tau)d\tau.$$

When $t\gg T_1$, the exponential term approaches zero, and

$$\int_0^t \psi(t)dt=\frac{}{\int_0^{T_1} G(\tau)d\tau}\int_0^{T_1}\tau G(\tau)d\tau \tag{8}$$

$$=\Sigma\eta/\Sigma G,$$

where $\Sigma\eta$ is the total viscosity of the system and ΣG the total stiffness. Similarly, it may be shown that

$$\int_0^t t\psi(t)dt=\frac{1}{\Sigma G}\int_0^{T_1}\tau^2 G(\tau)d\tau. \tag{9}$$

Then substituting Eqs. (8) and (9) into Eqs. (6) and (7), we have[4]

$$b=\frac{\int_0^{T_1} G(\tau)d\tau}{\int_0^{T_1}\tau G(\tau)d\tau} \tag{10}$$

and

$$a=\frac{\int_0^{T_1} G(\tau)d\tau\int_0^{T_1}\tau^2 G(\tau)d\tau}{\left[\int_0^{T_1}\tau G(\tau)d\tau\right]^2}. \tag{11}$$

Equations (10) and (11) define a and b in terms of the relaxation spectrum. Equations (6) and (7) are more useful, since they may be computed from primary data, while the distribution function $G(\tau)$ is derived by approximate methods from the relaxation function and other data. Further, Eq. (11) has in the numerator the second moment of this derived function, whereas (7) has only the first moment of an experimentally determined function. Both are highly sensitive to the form of the integrands at high values of t or τ, where precision is difficult to obtain, but the advantage seems to be with Eqs. (6) and (7).

Calculations of a and b were made on the IBM-650 digital computer, partly in conjunction with the calculation of $\varphi(t)$, using modified forms of Eqs. (6)and (7). A desk calculator was used separately with Eqs.

TABLE II. Parameters a and b of creep asymptote.

Equations	Method	a	b
(7) and (8)	Desk calculator	35 700	504/hr
(11) and (12)	Desk calculator	64 600	500/hr
(7) and (8)	IBM-650	34 676	496/hr

[3] C. Zener, *Elasticity and Anelasticity of Metals* (University of Chicago Press, Chicago, 1948).

[4] This is the usual expression for the steady-state creep rate, $V=F/\eta$, where V is strain rate, F is stress, and η is the total viscosity of all the relaxation mechanisms. It follows directly from physical consideration of the properties of the Maxwell array.

TABLE III. Creep function and creep asymptote.

Log time (hours)	Log creep function	Log creep asymptote
$\leqslant -0.3$	See Table I	4.54
−0.1	4.02	4.55
0.1	4.08	4.55
0.3	4.13	4.55
0.5	4.20	4.56
0.7	4.27	4.57
0.9	4.35	4.59
1.1	4.43	4.61
1.3	4.53	4.65
1.5	4.64	4.70
1.7	4.75	4.78
1.9	4.88	4.87
2.1	5.01	4.99
2.3	5.16	5.13

(10) and (11) as well as (6) and (7), using the Tobolsky and Catsiff[2] distribution function as well as the previously mentioned relaxation function. The results are given in Table II. The creep asymptote $\log[\varphi_a(t)] = \log(34\,676+496/\text{hr})$ is given with $\log\varphi(t)$ in Table III, and plotted with $\log\psi(t)$ and $\log\varphi(t)$ in Fig. 1.

DISCUSSION

It may be seen from Table III and Fig. 1 that the creep function comes into virtual coincidence with the asymptote at the last values calculated. The discrepancy between the values of a as calculated from the relaxation distribution time function and the stress relaxation function points to an inconsistency in these two functions at the longer times; even if the distribution function is considered to end at the last value given, the stress relaxation function would have finite though rapidly diminishing values at longer times. The greatest single incremental contributions in the numerical evaluations of a and b are given in Table IV. Here it will be seen that nearly 39% of the value 64 600 calculated for a by means of Eq. (11) comes from the last tabulated value of the relaxation time distribution function, the value which is probably the least precise of all the data. In any case, the great weight given to readings at the long-time end of the relaxation function indicates that attention to precision in these data is of great importance in calculations of the creep asymptote. To emphasize further this point, Table V gives the number of units of log time, starting from the long-time end, necessary to account for 90% and 99% of the calculated values of a and b.

TABLE IV. Maximum increments in numerical evaluations of creep asymptote parameters a and b.

Parameter	Equation	Log time for greatest increment	Ratio of greatest increment to total value of parameter
a	(7)	2.2, 2.4	0.217
b	(6)	2.0	0.116
a	(11)	2.6	0.387
b	(10)	2.2, 2.4, 2.6	0.126

TABLE V.

Parameter	Equation	Units of log time comprising 90% of parameter	Units of log time comprising 99% of parameter
a	(7)	0.6	0.9
b	(6)	1.2	1.8
a	(11)	0.5	0.8
b	(10)	0.9	1.4

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